On Pressure Perpendicular to the Shear Planes in Finite Pure Shears, and on the Lengthening of Loaded Wires when Twisted.

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In the 'Philosophical Magazine,' vol. 9, 1905, p. 397, I gave an analysis of the stresses in a pure shear which appeared to show that if ϵ is the angle of shear and if n is the rigidity, then a pressure $n\epsilon^2$ exists perpendicular to the planes of shear. That analysis is, I believe, faulty in that the diagonals of the rhombus into which a square is sheared are not the lines of greatest elongation and contraction, and are not at right angles after the shear, when second order quantities are taken into account, *i.e.*, quantities of the order of ϵ^2; I think the following analysis is more correct, and though it does not give a definite result, it leaves the existence of a longitudinal pressure an open question. The question appears to be answered in the affirmative by some experiments, described in the second part of the paper, in which loaded wires were found to lengthen when twisted by a small amount proportional to the square of the twist.

I.—*Stresses in a Pure Shear.*

Let a square ABCD (fig. 1) of side a be sheared into EFCD by motion through $AE = d$, the volume being constant. The angle of shear is $ADE = \epsilon$, and $\tan\epsilon = d/a$ exactly; neglecting ϵ^3, we may put $\epsilon = d/a$.

To find which line is stretched most by the shear, consider the line r drawn from D to P and making θ with DC before stretching.

Let it stretch to ρ, making θ' with DC; we have $r = a/\sin\theta$ and $\rho = a/\sin\theta'$;

also $$\rho^2 = r^2 + 2rd\cos\theta + d^2;$$

thus $$\begin{aligned}\rho^2/r^2 &= 1 + 2\,d/r\,.\cos\theta + d^2/r^2,\\ &= 1 + 2\,d/a\,.\sin\theta\cos\theta + d^2/a^2\,.\sin^2\theta.\end{aligned}$$

Differentiating ρ^2/r^2 with respect to θ, it is a maximum when

$$2\,d/a\,.\cos 2\theta + d^2/a^2\,.\sin 2\theta = 0,$$

or $$\tan 2\theta = -2\,a/d = -2\cot\epsilon.$$

Put $$\theta = 45^\circ + \delta, \qquad \text{then} \qquad \tan 2\delta = \tfrac{1}{2}\tan\epsilon,$$

or $\delta = \frac{1}{4}\epsilon$ to the second order, so that r makes $\frac{1}{4}\epsilon$ with the diagonal DB of the square through D, and on the upper side.

If the same shear is now made in the opposite direction ρ contracts to r, and the same directions of ρ before, and r after shear, give the maximum contraction. It is almost obvious that ρ makes $\frac{1}{4}\epsilon$ with DB on the lower side, but it may be verified by putting

$$r^2 = \rho^2 - 2\, d\rho \cos\theta' + d^2,$$

and finding the maximum value of r^2/ρ^2 after putting $\rho = a/\sin\theta'$ on the right.

Hence the lines of maximum elongation and contraction are at $\frac{1}{4}\epsilon$ with the diagonals of the square, and are at right angles before and after the strain, to the order of ϵ^2. It is noteworthy that as the shear increases the fibres which undergo maximum elongation and contraction change.

To find r and ρ put $r = a/\sin\theta = a/\sin(45^\circ + \frac{1}{4}\epsilon)$,

then $$r = \sqrt{2}\, a\,(1 - \tfrac{1}{4}\epsilon + \tfrac{3}{32}\epsilon^2)\,;$$

and changing the sign of ϵ we get

$$\rho = \sqrt{2}\, a\,(1 + \tfrac{1}{4}\epsilon + \tfrac{3}{32}\epsilon^2).$$

It is easily seen that the elongation and contraction are respectively

$$e = (\rho - r)/r = \tfrac{1}{2}\epsilon(1 + \tfrac{1}{4}\epsilon)\,; \qquad c = (r - \rho)/\rho = \tfrac{1}{2}\epsilon(1 - \tfrac{1}{4}\epsilon).$$

We shall now consider the stresses. We shall assume that a pressure P is put on in the direction of maximum contraction and a tension Q in the direction of maximum elongation, these being, as we have seen, at right angles; and we shall consider the equilibrium of the wedge ABC (fig. 2) when sheared, assuming that P and Q are the only forces on AC and BC. Let AB $= 2a$; AC $= \rho$; BC $= r$; these having the values just found.

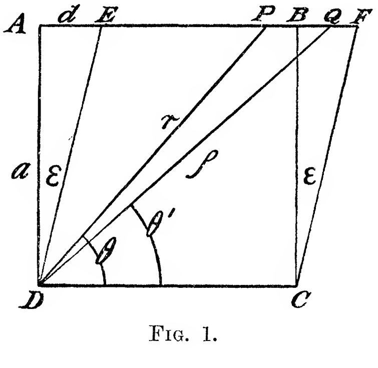

FIG. 1.

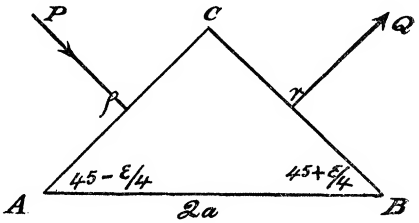

FIG. 2.

Resolve perpendicular to the base and let R be the pressure against the base, then

$$\begin{aligned} \mathrm{R}\,.\,2a &= \mathrm{P}\rho\cos(45^\circ-\tfrac{1}{4}\epsilon)-\mathrm{Q}r\cos(45^\circ+\tfrac{1}{4}\epsilon) \\ &= \mathrm{P}a\cot(45^\circ-\tfrac{1}{4}\epsilon)-\mathrm{Q}a\cot(45^\circ+\tfrac{1}{4}\epsilon) \\ &= \mathrm{P}a\frac{1+\frac{1}{4}\epsilon}{1-\frac{1}{4}\epsilon}-\mathrm{Q}a\frac{1-\frac{1}{4}\epsilon}{1+\frac{1}{4}\epsilon} \\ &= \mathrm{P}a(1+\tfrac{1}{2}\epsilon+\tfrac{1}{8}\epsilon^2)-\mathrm{Q}a(1-\tfrac{1}{2}\epsilon+\tfrac{1}{8}\epsilon^2) \\ &= (\mathrm{P}-\mathrm{Q})\,a+(\mathrm{P}+\mathrm{Q})\,.\,\tfrac{1}{2}a\epsilon+(\mathrm{P}-\mathrm{Q})\,.\,\tfrac{1}{8}a\epsilon^2, \end{aligned}$$

where P and Q can only be taken as equal to the first order. Proceeding to the second order, we must put

$$\mathrm{P} = n\epsilon+p\epsilon^2\,;$$

then

$$\mathrm{Q} = n\epsilon-p\epsilon^2,$$

where p is a constant to the second order.

Thus

$$\mathrm{P}-\mathrm{Q} = 2p\epsilon^2 \quad\text{and}\quad \mathrm{P}+\mathrm{Q} = 2n\epsilon,$$

and

$$\mathrm{R} = (\tfrac{1}{2}n+p)\,\epsilon^2,$$

the third term being negligible. If we resolve parallel to the base, it is easily found that the tangential stress is

$$\mathrm{T} = \tfrac{1}{2}(\mathrm{P}+\mathrm{Q}) = n\epsilon.$$

If the shear is produced by a tangential stress T, then it requires the system P, Q, and R to maintain equilibrium with it.

It is possible that a stress exists perpendicular to the plane of the figure in fig. 1. It can only be assumed that the changes of dimension in that direction neutralise each other to the first order when equal pushes and pulls are put on in the plane of the figure; when the dimensions perpendicular to the figure are constrained to remain the same to the second order—and this is our supposition—it may require a tension or pressure to effect this.

Let us suppose that a pressure $\mathrm{S} = q\epsilon^2$ is introduced, a tension if q is negative. To make $\mathrm{R} = 0$ we should require to have $p = -\frac{1}{2}n$; also P would then be less than Q. If pressure perpendicular to AC is exerted alone, and then tension perpendicular to BC is exerted alone, it appears probable that for very large equal compressions and extensions P is greater than Q. If we suppose that when they are simultaneous the tendency is in the same direction, then R should have a positive value, or the longitudinal pressure perpendicular to AB should exist.

Let us examine the consequences of the supposition that both R and S exist. Let a thin tube of length l and of radius a be fixed at one end, and let the other end be twisted through θ so that the angle of shear is $\epsilon = a\theta/l$. Let an end pressure $\mathrm{R} = (\frac{1}{2}n+p)\epsilon^2$ be put on, and also a side pressure $\mathrm{S} = q\epsilon^2$ so as to maintain constant dimensions.

The side pressure S may be replaced by a uniform pressure S over the whole surface, and a tension S over the ends. We have then an end pressure R−S and a pressure S all over.

Now suppose that these forces are removed. Through the removal of R−S we shall have a lengthening dl_1 given by

$$(\tfrac{1}{2}n+p-q)\,\epsilon^2 = \mathrm{Y}\,dl_1/l,$$

and a contraction δ_1 of the diameter given by

$$\delta_1/2\,a = \sigma\,dl_1/l,$$

where Y is Young's modulus and σ is Poisson's ratio.

Through the removal of the pressure S we shall have a lengthening dl_2 given by $q\epsilon^2 = 3\,\mathrm{K}dl_2/l$, where K is the bulk modulus, and an expansion δ_2 of the diameter given by $q\epsilon^2 = 3\mathrm{K}\delta_2/2a$.

The end lengthening is therefore

$$dl = dl_1+dl_2 = \{(\tfrac{1}{2}n+p-q)/\mathrm{Y}+q/3\,\mathrm{K}\}\,l\epsilon^2;$$

or putting $$1/3\,\mathrm{K} = 3/\mathrm{Y}-1/n,$$

it is $$\{(\tfrac{1}{2}n+p)/\mathrm{Y}+(2/\mathrm{Y}-1/n)\,q\}\,l\epsilon^2 = sl\epsilon^2 = sa^2\theta^2/l,$$

where s is put for $$(\tfrac{1}{2}n+p)/\mathrm{Y}+(2/\mathrm{Y}-1/n)\,q.$$

The diameter decreases by

$$\begin{aligned}\delta = \delta_1-\delta_2 &= \{(\tfrac{1}{2}n+p-q)\,\sigma/\mathrm{Y}-q/3\,\mathrm{K}\}\,2\,a\epsilon^2,\\ &= \{(\tfrac{1}{2}n+p-q)\,\sigma/\mathrm{Y}-[(3+\sigma)\,\mathrm{Y}-1/n]\,q\}\,2\,a^3\theta^2/l^2.\end{aligned}$$

It would not be easy to test this result with a thin tube. But if we suppose that a wire extends by the amount equal to the average extension of the tubes into which it may be resolved, we get

$$dl = \frac{1}{\pi a^2}\int_0^a \frac{2\,\pi r s r^2\theta^2}{l}\,dr = \tfrac{1}{2}\,sa^2\theta^2/l.$$

I now proceed to describe some experiments which show that such an extension exists.

II.—*The Lengthening of Loaded Wires when twisted.*

Experiments were made on several wires hung vertically from a fixed support, and loaded in order that kinks or remnants of the spiral due to the coiling to which they had been subjected might be taken out. This was considered to be effected when the stretch for a given addition of load was sensibly the same whether the wire was twisted or not. An account of the twisting of a steel wire before this stage was reached will be given later. Fig. 3 represents the arrangement more or less diagrammatically. The upper end of the wire to be twisted was fixed to a stout bracket B near the ceiling. The wire was always about 231 cm. long; its lower end was clamped in jaws in the upper end of a turned steel rod *rr*, 51 cm. long, which passed through a hole in the table T on which was the observing microscope M, and a parallel plate micrometer *m*. One division of this micrometer was equal to 0·00974 mm. At the lower end of the rod was a horizontal iron cross-piece *cc*, 19 cm. long and 1·6 cm. square. From the lower end was suspended a carrier for the weights, or for the two stouter wires the weight itself, connected to the rod by a flat steel strip twisted in its middle, so that the upper and lower halves were in two vertical planes at right angles. Below the weights was a set of vanes immersed in a shallow bath of oil to damp vibrations. This bath rested on a circular turntable *tt*, on which were two uprights *uu* at opposite ends of a diameter, with horizontal screws at their upper ends which could be brought to bear against the ends of the cross-piece as shown in the plan, fig. 4. The screws ended in small steel balls and the sides of the cross-piece were polished. On rotating the turntable the screws came against the cross-piece and turned it round; and so the wire was twisted by a couple with vertical axis. The axis of the turntable was made vertical by means of the levelling screws *ll*. To adjust this axis in the axis of the wire prolonged, the turntable could be moved over the base plate P by means of the horizontal screws *s*, of which only one is represented in fig. 3. All are shown in fig. 4. A horizontal microscope not represented in the figure was attached to one of the uprights and focussed on the edge of the rod *rr*. The adjustment by the screws *s* was

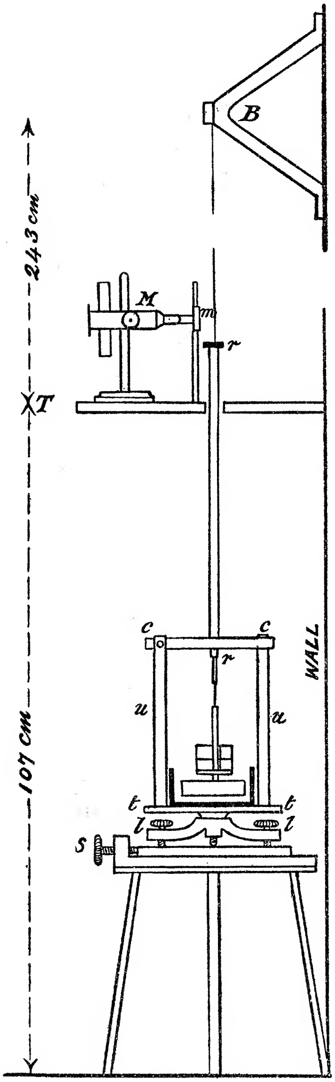

Fig. 3.—Elevation of Arrangement for Twisting the Wires.

continued until the microscope always saw the edge of the rod in the middle of the field, however the turntable might be turned.

To give a definite point of view in the microscope M, in the earlier experiments starch grains were put on the wire about 1 cm. from the lower end. These were illuminated, and a suitable one was selected.

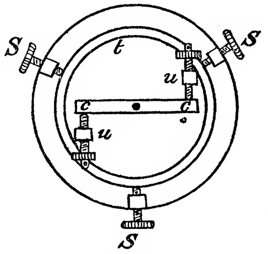

Fig. 4.—Plan of the Cross-piece and Turntable.

In the later experiments a needle, about 1 cm. long, was fixed on the upper end of the rod, point upwards, close alongside the wire, and the needle point was viewed. This was better than the starch grains.

In the earlier work the temperature of the room was fairly steady, and the changes in length due to temperature variations were too slow to give trouble. But in some gusty weather occurring later there were such rapid and considerable variations in the temperature of the room that it was necessary to enclose the wire in a wooden tube. After this was done temperature gave no further trouble, whatever the weather.

In order to observe the effect of a twist the turntable was levelled and adjusted axially when the wire and cross-piece were free. The turntable was rotated till the screws on the uprights just touched the cross-piece. Then chalk marks were made on the turntable and on the plate below, one just over the other. The microscope was adjusted exactly to sight the upper or lower edge of a starch grain on its horizontal cross-wire, and the micrometer was read. Then the turntable was rotated so many whole turns, and the micrometer plate was moved till the edge of the grain was again on the cross-wire and the micrometer was read again.

With the exception of a wire stretched only by the weight of the rod and cross-piece, in some experiments described later, there was always a lengthening on twisting, of the same order whether the twist was clockwise or counter clockwise. The lengthening was nearly proportional to the square of the twist put on. It was necessary to limit the twist to a few turns to avoid permanent set, and when such a small twist had been given and the wire was untwisted it returned sensibly to its original length.

The lowering was entirely due to twisting and not to any giving of the support, for when a microscope was sighted on a point on the wire close to the upper end, no change in level could be detected, when the wire was twisted through 5 turns at its lower end. This was further verified by an experiment on a steel wire from the same piece as No. 3 below, which showed that the extension half-way down the wire was, within the limits of experimental error, half that at the lower end. A microscope and micrometer were fixed on a table half-way up the wire and a needle point was fixed here as well as at the lower end. At each twist and untwist both micrometers were read. I give the observations in this experiment in full, as they will show the sort of accuracy attained.

The lower end was twisted from a starting twist of ¼ turn to 4¼ turns.

Micrometer Readings at Lower End.

¼ turn.	4¼ turns.	Lowering.
22·3	18·6	3·7
22·5	19·0	3·5
23·0	19·2	3·8
22·6	19·4	3·2
22·9	19·6	3·3
22·6	19·5	3·1
23·0	19·5	3·5
23·0	18·9	4·1
22·7	19·2	3·5
22·9	19·6	3·3

Mean lowering, 3·50 divisions.
One division of micrometer = 0·00974 mm.
The lowering is 0·0341 mm.

Micrometer Readings Half-way up the Wire.

¼ turn.	4¼ turns.	Lowering.
30·4	28·3	2·1
30·5	28·4	2·1
30·2	28·1	2·1
30·5	28·5	2·0
30·5	28·0	2·5
30·4	28·7	1·7
31·0	28·4	2·6
30·6	28·5	2·1
31·8*	29·9	1·9
31·6*	29·9	2·5

* Another point on the needle sighted.

Mean lowering, 2·16 divisions.
One division of micrometer = 0·00751 mm.
The lowering is 0·0162 mm.

If the lowering at the end is accurate, that half-way up should be 0·0171 mm. The observed lowering is as nearly equal to this as could be expected.

With the first wire, determinations of extension due to an addition of 520 grammes were made both in the untwisted and twisted conditions, as it was only when these became sensibly equal that the lowering on twist became equal for different loads. The extra load could be put on or taken off by lowering or raising a lever, not represented in fig. 3. It is unnecessary to describe the details of this arrangement. The experiments with the other wires were made with such loads that it was not considered necessary to observe the stretch due to addition of load.

Results.

1*a*. Steel piano wire, diameter 0·720 mm. (mean of 10 measurements at different points), length in this and all cases to observing point 230 cm. Permanent set after putting on eight turns twist and then untwisting only a very few degrees. Total load, 7081 grammes.

The twist is termed clockwise when the turntable as viewed from above is moved clockwise.

Clockwise twist, 0—4 turns; lowering 0·0181 mm., mean of 10 observations.
" " 0—8 " " 0·0732 " " "

The ratio of these is 4·04 : 1.

The extension due to an addition of 520 grammes was:—

No twist on the wire 0·143 mm., mean of 10 observations.
4 turns " 0·141 " " "
8 " " 0·143 " " "

1*b*. Same wire.

Total load, 9081 grammes.

Clockwise twist, 0—4 turns; lowering 0·0181 mm., mean of 20 observations.
" " 0—8 " " 0·0749 " " "

The ratio of these is 4·14 : 1.

The extension due to an addition of 520 grammes was:

No twist on the wire 0·142 mm., mean of 10 observations.
8 turns " 0·144 " " "

Taking the mean lowering for the two loads of 7081 and 9081 for eight turns twist, viz., 0·074 mm., and taking it as proportional to the square of the twist, the lowering for one turn is 0·00116 mm., and

$$s = 2\,l\,dl/a^2\theta^2 = 1{\cdot}043.$$

The moduli of elasticity of this wire were found to be

$$n = 0{\cdot}769\times 10^{12}, \qquad \mathrm{Y} = 2{\cdot}013\times 10^{12},$$

whence $n/\mathrm{Y} = 0{\cdot}382$. The value of n, found for loads of 1081 grammes and 9081 grammes respectively, was identical.

2. The same wire was raised to a red heat, by an electric current, with the load of 9081 grammes on it. It lengthened about 3 cm., and this length was cut off. The surface oxidised, and when the oxide was rubbed off the diameter was 0·696 mm. (mean of 10).

The permanent set after twisting and untwisting was greater, and so only three turns were given.

Total load, 9081 grammes.

Clockwise twist, 0—3 turns; lowering 0·0129 mm., mean of five observations.

The extension due to an addition of 520 grammes was:—

No twist on the wire 0·155 mm., mean of 10 observations.
3 turns „ 0·154 „ „ „

The lowering for one turn according to the square law is: 0·00143 mm., and $s = 1·376$ mm.

The five values of the lowering were: 1·5, 1·2, 1·2, 1·2, 1·5 divisions, mean 1·32 divisions. With such small lowering no accuracy could be expected, and it would be difficult to verify the square law.

The moduli of elasticity for the softened wire were:—

$$n = 0·809 \times 10^{12} \quad \text{and} \quad Y = 2·06 \times 10^{12},$$

whence

$$n/Y = 0·393.$$

3. Steel piano wire, diameter 0·970 mm. (mean of 10).

A needle point fixed at the side of the wire was viewed in the microscope. After twisting and untwisting, a slight permanent set threw the point out of focus if the start was from no twist. A quarter turn was therefore put on initially, and the twisting was from this, and the untwisting was back to it.

Total load, 19,504 grammes.

Clockwise twist, $\frac{1}{4}$—$2\frac{1}{4}$ turns; lowering 0·0087 mm., mean of 10 observations.
„ „ $\frac{1}{4}$—$4\frac{1}{4}$ „ „ 0·0339 „ „ „
Counter clockwise twist, $\frac{1}{4}$—$2\frac{1}{4}$ turns; lowering 0·0089 mm., mean of 10 obs.
„ „ „ $\frac{1}{4}$—$4\frac{1}{4}$ „ „ 0·0340 „ „

Mean lowering, $\frac{1}{4}$—$2\frac{1}{4}$, 0·0088 mm.
„ „ $\frac{1}{4}$—$4\frac{1}{4}$, 0·0340 „

By the square law the lowerings for $4\frac{1}{4}$, $2\frac{1}{4}$, and $\frac{1}{4}$ should be as 289 : 81 : 1, and the differences should be as 288 : 80 = 18 : 5.

The observed differences are as 18·9 : 5.

The lowering for one turn deduced from the difference between $\frac{1}{4}$ and $4\frac{1}{4}$ is 0·00189 mm., and $s = 0·936$.

Comparing the lowerings for one turn of this wire with the hard wire No. 1, if the lowering is proportional to the square of the diameter, we ought to have for No. 1 a lowering of $0{\cdot}00189 \times (72/97)^2 = 0{\cdot}00104$ mm. The observed lowering was 0·00116 mm., which is as near the calculated value as could be expected.

4. The same wire was then raised to a red heat by an electric current with the load on. After being rubbed down its diameter was 0·947 mm. (mean of 10 measurements). Same load as 3.

Clockwise twist, ¼—3¼ turns, lowering 0·0207 mm., mean of 10 observations.

The deduced lowering for one turn is 0·00197 mm.

The value of s is 1·017 mm.

Comparing the lowerings for one turn of this wire with the softened wire No. 2, the square law for the diameter should give for No. 2 a lowering $0{\cdot}00197 \times (696/947)^2 = 0{\cdot}00106$ mm. The observed lowering was 0·0143 mm., a considerable divergence.

5. Copper wire, diameter 0·655 mm. (mean of 10). Load, 7081 grammes.

Clockwise twist, ¼—2¼ turns; lowering 0·0066 mm., mean of 10 observations.
Counter clockwise twist, ¼—2¼ turns; lowering 0·0083 mm., mean of 10 obs.

It was not safe to give a greater twist owing to the largeness of the permanent set. With 2¼ turns the set was still small.

The larger value of the lowering for the counter clockwise twist is almost certainly real, and not merely error of observation. Some other observations showed an even greater excess, though they were very irregular owing to temperature variations, and are not worth recording. The extension due to an addition of 520 grammes was:—

No twist on the wire 0·268 mm., mean of 20 observations.
3 turns „ 0·269 „ „ „

Taking the mean for clockwise and counter clockwise twist, the lowering for one turn is 0·0015 mm, and $s = 1{\cdot}63$ mm.

6. Brass wire, diameter 0·928 mm. (mean of 10). Load 19,504 grammes.

Clockwise twist, ¼—2¼ turns; lowering 0·0169 mm., mean of 10 observations.
„ „ ¼—4¼ „ „ 0·0540 „ „ „
Counter clockwise twist, ¼—2¼ turns; lowering 0·0135 mm., mean of 10 obs.
„ „ „ ¼—4¼ „ „ 0·0479 „ „

The difference between clockwise and counter clockwise twisting is too large for errors of observation.

For the square law the lowerings for ¼—4¼, and for ¼—2¼ should be in the ratio 18 : 5. They are in the ratios 16 : 5 for clockwise, and 17·7 : 5 for counter clockwise twisting. The lowering for one turn clockwise, as deduced from $\frac{1}{4}$—$4\frac{1}{4}$, is 0·0030 mm., and for one turn counter clockwise is 0·0026 mm.

The mean value of $s = 1·537$ mm.

Experiments with Smaller Loads.

When the piano wire diameter 0·72 mm. was loaded only with the rod and cross bar weighing 1081 grammes, there was a *rise* on twisting.

Clockwise twist,	0—4	turns, rise	0·041 mm.
„ „	0—8	„	0·139 „
Counter clockwise twist,	0—4	„	0·023 „
„ „ „	0—8	„	0·108 „

The extension for an addition of 520 grammes was:—

No twist on wire	0·137	mm., mean of	6	observations.
4 turns clockwise	0·170	„ „	6	„
8 „ „	0·237	„ „	3	„
4 turns counter clockwise	0·156	„ „	6	„
8 „ „	0·238	„ „	3	„

If by means of the observed extensions we calculate the positions of the point viewed, when the load of 1081 grammes is taken off we find that the total rise would be for clockwise twist: for four turns 0·110 mm., and for eight turns, 0·347 mm.

The rise appears to be due to coiling up of the wire on twisting, through some remnant of the spiral condition in which it existed before suspension. This is confirmed by the very large increase in extension, due to addition of load as the twist on the wire is increased. It may be a coincidence that the rise on twisting and the increase of stretch are both nearly proportional to the square of the number of turns.

Experiments were then made with greater loads to find how the lowering and extension changed. Only clockwise twist was observed.

Load 3081 grammes, the rise changed to lowering.

Twist,	0—4 turns;	lowering	0·0131	mm., mean of 20 observations.
„	0—8 „	„	0·0498	„ „ „

The extension due to an addition of 520 grammes was:—

No twist on the wire	0·144	mm., mean of	10 observations.
4 turns „	0·143	„ „	„
8 „ „	0·149	„ „	„

Showing a still slight excess of extension in the most twisted condition.

Load 5081 grammes.

Twist,	0—4 turns;	lowering	0·0164	mm., mean of 20 observations.
„	0—8 „	„	0·0660	„ „ „

The extension due to an addition of 520 grammes was:—

No twist on the wire 0·141 mm., mean of 10 observations.
4 turns „ 0·142 „ „ 15 „
8 „ „ 0·144 „ „ 15 „

The results for loads of 7081 and 9081 grammes are already recorded under 1*a* and 1*b*. There is obviously a tendency for the lowering to increase with load until the extensions under different twists become more nearly equal with equal added load.

When the same wire was softened and loaded with 3081 grammes the lowering for three turns was 0·0093 mm. (mean of 10 observations).

The extension due to an addition of 520 grammes was:—

No twist on the wire 0·149 mm., mean of 10 observations.
3 turns „ 0·147 „ „ „

With load 9081 the same wire gave the results recorded under 2, which show a greater lowering for an equal twist but the same extension with added load.

The copper wire diameter 0·655 mm. (No. 5 above) with load 4081 grammes gave:—

Clockwise twist, 0—3 turns, 0·00965 mm., mean of 10 observations.
Counter clockwise twist, 0—3 turns, 0·0156 mm., mean of 10 observations.

Taking the mean of these, the lowering for one turn is 0·0014 mm.

The extension due to an addition of 520 grammes was:—

No twist on the wire 0·268 mm., mean of 5 observations;
3 turns „ 0·270 „ „ „

extensions agreeing very nearly with those recorded above for a load of 6081 grammes on the same wire.

Remarks on the Results of Measurements.

The lowering was never so much as 0·1 mm. and was usually much less. The accuracy attained could hardly be expected to be great. The measurements, however, appear to show that when a wire is sufficiently loaded to be straightened, it is lengthened by twisting by an amount proportional to the square of the twist and with a given number of turns inversely as the length.

It might be thought possible that the effect observed was due to rise of temperature, either through adiabatic strain or through dissipation of strain energy as heat. But the observations give no support to this explanation. When the wire was extended by twisting, it remained extended, and when untwisted it returned. Temperature effects would be a maximum the instant after twisting, and would then gradually subside. It may be noted that the adiabatic change of temperature is proportional to $a^2\theta^2/l$, but it is a cooling, and its amount is such as to shorten the wire, in the case of steel, by something of the order of 1/100 of the observed extension. If we suppose that some definite fraction of the strain energy put in is dissipated, again the change, now a warming, is proportional to $a^2\theta^2/l$. The whole strain energy, in the case of steel, would only raise the temperature by an amount accounting for something of the order of 1/10 the observed extension, and, in fact, only an exceedingly minute fraction of the strain energy is dissipated.

A comparison of the wires (1) and (3) appears to show that the lengthening for a given twist is proportional to the square of the radius.

If we put the lengthening

$$dl = sa^2\theta^2/2l,$$

s for steel is in the neighbourhood of 1. For copper and brass s is in the neighbourhood of 1·5. The lowering for the copper and brass wires tested for twists in opposite directions is not the same.

With a hard steel wire with small load the end of the wire *rises* on twisting, probably through coiling.

The value of $s = (\frac{1}{2}n+p)/\mathrm{Y}-(2/\mathrm{Y}-1/n)q$ appears to be measurable, but its value gives us no clue to the values of p and q.

If we could assume $q = 0$, then for steel we should have p about $2n$, but I see no justification for the assumption.

If we could measure the decrease in diameter, we should obtain the value of $(\frac{1}{2}n+p-q)\sigma/\mathrm{Y}-\{(3+\sigma)/\mathrm{Y}-1/n\}q$, and knowing n, Y, and σ we should be able to find p and q. But a thin wire is quite unsuitable for this measurement. The decrease is probably of the order of $2\,a/l \times$ lengthening. With the wires I have used this is of the order 1/1000 × lengthening, and an accuracy of measurement of 10^{-6} mm. would be required at least. With a shaft of considerable diameter it might be possible to measure the quantity, though the experimental difficulties are obviously very great.

The Effect of the Lengthening of a Wire on its Torsional Vibration.

If a wire is loaded with mass M having moment of inertia I, when M is set vibrating torsionally it falls and rises as it swings, its distance below the highest point being given by

$$x = \tfrac{1}{2}sa^2\theta^2/l.$$

The kinetic energy is $\mathrm{T} = \frac{1}{2}\mathrm{I}\dot{\theta}^2+\frac{1}{2}\mathrm{M}\dot{x}^2.$

The last term is easily found to be negligible.

The potential energy is

$$V = \tfrac{1}{4}\, n\pi a^4\theta^2/l - Mgx = \tfrac{1}{4}\, n\pi a^4\theta^2/l - \tfrac{1}{2}\, Mgsa^2\theta^2/l.$$

The equation of motion is

$$I\ddot{\theta} + (\tfrac{1}{2}\, n\pi a^4/l - Mgsa^2/l)\,\theta = 0.$$

Whence

$$T^2 = \frac{8\,\pi^2 Il}{n\pi a^4(1 - 2\,Mgsa^2/n\pi a^4)},$$

and T is greater than it would be if s were 0 by the factor

$$1 + Mgs/n\pi a^2.$$

If Y is Young's modulus and if e is the elongation of the wire due to the load Mg,

$$Mg/\pi a^2 = Ye,$$

so that the factor may be conveniently written as $1 + se\,Y/n$.

If the vibrations are used to determine the modulus of rigidity n, then the value of n will be greater than that deduced by neglect of s, and by the factor $1 + 2\,se\,Y/n$.

To give an idea of the effect on the determination of the modulus of rigidity, let us suppose that a quite straight steel wire, diameter 0·7 mm., has a load of 2000 grammes. For steel Y/n is about 2·6. For the given diameter e is about 2×10^{-4}. We have found that s is about 1. The correcting factor is then about 1·001, or the true rigidity exceeds the value calculated in the ordinary way by about 1 in 1000. If the wire is not sufficiently loaded to be straight the value of s is less. If very lightly loaded the sign of s may be changed and the true rigidity may be less than the value as ordinarily calculated. The correction is hardly needful in practice, as the modulus of rigidity is probably not measurable to three figures.

Distortional Waves.

In purely distortional waves in a medium of great extent it is evident that the pressure S perpendicular to the axes of shear, if it exists, will not produce any motion. To keep the waves purely distortional, *i.e.* with motion perpendicular to the direction of propagation only, a force must be applied from outside dR/dx per cubic centimetre in the direction of propagation. If this force is not applied then longitudinal motion must result, obviously of the second order, unless $\tfrac{1}{2}\, n + p = 0$. This is probably the condition for an incompressible medium. If $\tfrac{1}{2} n + p$ is not zero it appears possible that dispersion may exist. If the longitudinal motion is neglected the pressure in the direction of propagation is $(\tfrac{1}{2}\, n + p)\,\epsilon^2$ and all that we can say, at present, is that it is probably of the order of $n\epsilon^2$.